

# MIRROR SYMMETRY FOR CALABI-YAU COMPLETE INTERSECTIONS IN FANO TORIC VARIETIES

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ABSTRACT. Generalizing the notions of reflexive polytopes and nef-partitions of Batyrev and Borisov, we propose a mirror symmetry construction for Calabi-Yau complete intersections in Fano toric varieties.

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## 0. INTRODUCTION.

Reflexive polytopes  $\Delta$  in  $\mathbb{R}^d$ , introduced by Victor Batyrev in [B1], are determined by the property that they have vertices in  $\mathbb{Z}^d$  and have the origin in their interior with the polar dual polytope

$$\Delta^* = \{y \in \mathbb{R}^d \mid \langle \Delta, y \rangle \geq -1\}$$

satisfying the same property. The polar duality gives an involution between the sets of reflexive polytopes:  $(\Delta^*)^* = \Delta$ . There is a one-to-one correspondence between isomorphism classes of reflexive polytopes  $\Delta$  in  $\mathbb{R}^d$  and  $d$ -dimensional Gorenstein Fano toric varieties given by

$$\Delta \mapsto X_\Delta := \text{Proj}(\mathbb{C}[\mathbb{R}_{\geq 0}(\Delta, 1) \cap \mathbb{Z}^{d+1}]),$$

where the grading is induced by the last coordinate in  $\mathbb{Z}^{d+1}$ . The dual pair of reflexive polytopes  $\Delta$  and  $\Delta^*$  corresponds to the *Batyrev mirror pair* of ample Calabi-Yau hypersurfaces  $Y_\Delta \subset X_\Delta$  and  $Y_{\Delta^*} \subset X_{\Delta^*}$  in Gorenstein Fano toric varieties in [B1]. By taking maximal projective crepant partial resolutions  $\hat{Y}_\Delta \rightarrow Y_\Delta$  and  $\hat{Y}_{\Delta^*} \rightarrow Y_{\Delta^*}$  induced by toric blow ups, Batyrev obtained a mirror pair of minimal Calabi-Yau hypersurfaces  $\hat{Y}_\Delta, \hat{Y}_{\Delta^*}$ .

Generalizing the polar duality of reflexive polytopes, Lev Borisov in [Bo] introduced the notion of *nef-partition*, which is a Minkowski sum decomposition of the

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reflexive polytope  $\Delta = \Delta_1 + \cdots + \Delta_r$  by lattice polytopes such that the origin  $0 \in \Delta_i$  for all  $1 \leq i \leq r$ . A nef-partition has a *dual nef-partition* defined as the Minkowski sum decomposition of the reflexive polytope  $\nabla = \nabla_1 + \cdots + \nabla_r$  in the dual vector space with  $\nabla_j$  determined by  $\langle \Delta_i, \nabla_j \rangle \geq -\delta_{ij}$  for all  $1 \leq i, j \leq r$ , where  $\delta_{ij}$  is the Kronecker symbol.

A nef-partition of a reflexive polytope  $\Delta = \Delta_1 + \cdots + \Delta_r$  with  $r < d$  and  $\dim \Delta_i > 0$ , for all  $1 \leq i \leq r$ , defines a nef Calabi-Yau complete intersection  $Y_{\Delta_1, \dots, \Delta_r}$  in the Gorenstein Fano toric variety  $X_\Delta$  given by the equations:

$$\left( \sum_{m \in \Delta_i \cap \mathbb{Z}^d} a_{i,m} \prod_{v_\rho \in \mathcal{V}(\Delta^*)} x_\rho^{\langle m, v_\rho \rangle} \right) \prod_{v_\rho \in \nabla_i} x_\rho = 0, \quad i = 1, \dots, r,$$

with generic  $a_{i,m} \in \mathbb{C}$ , where  $x_\rho$  are the Cox homogeneous coordinates of the toric variety  $X_\Delta$  corresponding to the vertices  $v_\rho$  of the polytope  $\Delta^*$ .

The Batyrev-Borisov mirror symmetry construction is a pair of families of generic nef Calabi-Yau complete intersections  $Y_{\Delta_1, \dots, \Delta_k} \subset X_\Delta$  and  $Y_{\nabla_1, \dots, \nabla_k} \subset X_\nabla$  in Gorenstein Fano toric varieties corresponding to a dual pair of nef-partitions  $\Delta = \Delta_1 + \cdots + \Delta_r$  and  $\nabla = \nabla_1 + \cdots + \nabla_r$ . By taking maximal projective crepant partial resolutions  $\hat{Y}_{\Delta_1, \dots, \Delta_k} \rightarrow Y_{\Delta_1, \dots, \Delta_k}$  and  $\hat{Y}_{\nabla_1, \dots, \nabla_k} \rightarrow Y_{\nabla_1, \dots, \nabla_k}$ , one obtains the *Batyrev-Borisov mirror pair* of minimal Calabi-Yau complete intersections.

The topological mirror symmetry test for compact  $n$ -dimensional Calabi-Yau manifolds  $V$  and  $V^*$  is a symmetry of their Hodge numbers:

$$h^{p,q}(V) = h^{n-p,q}(V^*), \quad 0 \leq p, q \leq n.$$

For singular varieties Hodge numbers must be replaced by the stringy Hodge numbers  $h_{\text{st}}^{p,q}$  introduced by V. Batyrev in [B2]. The usual Hodge numbers coincide with the stringy Hodge numbers for nonsingular Calabi-Yau varieties, and all crepant partial resolutions  $\hat{V}$  of singular Calabi-Yau varieties  $V$  have the same stringy Hodge numbers:  $h_{\text{st}}^{p,q}(\hat{V}) = h_{\text{st}}^{p,q}(V)$ . In [BBo2], Batyrev and Borisov show that the pair of generic Calabi-Yau complete intersections  $V = Y_{\Delta_1, \dots, \Delta_r}$  and  $V^* = Y_{\nabla_1, \dots, \nabla_r}$  pass the topological mirror symmetry test:

$$h_{\text{st}}^{p,q}(Y_{\Delta_1, \dots, \Delta_r}) = h_{\text{st}}^{d-r-p,q}(Y_{\nabla_1, \dots, \nabla_r}), \quad 0 \leq p, q \leq d-r.$$

Generalizing the notions of reflexive polytopes and nef-partitions for rational polytopes we introduce the notions of  $\mathbb{Q}$ -reflexive polytopes and  $\mathbb{Q}$ -nef-partitions. A  $\mathbb{Q}$ -reflexive polytope  $\Delta$  in  $\mathbb{R}^d$  is determined by the properties that 0 is in the interior of  $\Delta$  and

$$\text{Conv}((\text{Conv}(\Delta \cap \mathbb{Z}^d))^* \cap \mathbb{Z}^d) = \Delta^*.$$

We show that a  $\mathbb{Q}$ -reflexive polytope  $\Delta$  corresponds to the Fano toric variety  $X_\Delta$  with at worst canonical singularities. A  $\mathbb{Q}$ -reflexive polytope has a *dual  $\mathbb{Q}$ -reflexive polytope* defined by  $\Delta^\circ := (\text{Conv}(\Delta \cap \mathbb{Z}^d))^*$  and the property  $(\Delta^\circ)^\circ = \Delta$  gives an involution on the set of  $\mathbb{Q}$ -reflexive polytopes. The dual lattice polytope  $\Delta^*$  of a  $\mathbb{Q}$ -reflexive polytope is called an *almost reflexive polytope* and there is a similar involution on the set of almost reflexive polytopes. All reflexive polytopes are  $\mathbb{Q}$ -reflexive and almost reflexive.

A  $\mathbb{Q}$ -nef-partition is a Minkowski sum decomposition  $\Delta = \Delta_1 + \cdots + \Delta_r$  of a  $\mathbb{Q}$ -reflexive polytope into polytopes in  $\mathbb{R}^d$  such that  $0 \in \Delta_i$  for all  $1 \leq i \leq r$ , and

$$\text{Conv}(\Delta \cap \mathbb{Z}^d) = \text{Conv}(\Delta_1 \cap \mathbb{Z}^d) + \cdots + \text{Conv}(\Delta_r \cap \mathbb{Z}^d).$$

We prove that a  $\mathbb{Q}$ -nef-partition  $\Delta = \Delta_1 + \cdots + \Delta_r$  has a *dual  $\mathbb{Q}$ -nef-partition*  $\nabla = \nabla_1 + \cdots + \nabla_r$  determined by  $\langle \Delta_i \cap \mathbb{Z}^d, \nabla_j \rangle \geq -\delta_{ij}$  for all  $1 \leq i, j \leq r$ . This dual pair of  $\mathbb{Q}$ -nef-partitions corresponds to a pair of  $\mathbb{Q}$ -nef Calabi-Yau complete intersections  $Y_{\Delta_1, \dots, \Delta_k} \subset X_\Delta$  and  $Y_{\nabla_1, \dots, \nabla_k} \subset X_\nabla$  in Fano toric varieties. We expect that this pair passes the topological mirror symmetry test as in [BBo2].

In [BBo1], Batyrev and Borisov introduce the notion of reflexive Gorenstein cones  $\sigma \subset \mathbb{R}^{\bar{d}}$ , which canonically correspond to Gorenstein Fano toric varieties  $X_\sigma = \text{Proj}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^{\bar{d}}])$  such that  $\mathcal{O}_{X_\sigma}(1)$  is an ample invertible sheaf and there is a positive integer  $r$  such that  $\mathcal{O}_{X_\sigma}(r)$  is isomorphic to the anticanonical sheaf of  $X_\sigma$ . The zeros  $Y_\sigma$  of generic global sections of  $\mathcal{O}_{X_\sigma}(1)$  are called *generalized Calabi-Yau manifolds*. The dual cone

$$\sigma^\vee = \{y \in \mathbb{R}^{\bar{d}} \mid \langle x, y \rangle \geq 0 \forall x \in \sigma\}$$

of a reflexive Gorenstein cone  $\sigma$  is again reflexive, and the dual pair  $\sigma$  and  $\sigma^\vee$  corresponds to the *mirror pair* of generalized Calabi-Yau manifolds  $Y_\sigma$  and  $Y_{\sigma^\vee}$ , which are ample hypersurfaces in the respective Gorenstein Fano toric varieties.

Combining the ideas of [BBo1] with the notion of almost reflexive polytopes, we introduce the notion of *almost reflexive* Gorenstein cones  $\sigma$ . Their dual cones  $\sigma^\vee$  are no longer Gorenstein, but there is a canonically defined grading on  $\sigma^\vee \cap \mathbb{Z}^{\bar{d}}$ . This allows us to associate to an almost reflexive Gorenstein cone  $\sigma \subset \mathbb{R}^{\bar{d}}$  the Fano toric variety  $X_\sigma = \text{Proj}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^{\bar{d}}])$ . The reflexive rank one sheaf  $\mathcal{O}_{X_\sigma}(1)$  corresponds to an ample  $\mathbb{Q}$ -Cartier divisor and there is a positive integer  $r$  such that  $\mathcal{O}_{X_\sigma}(r)$  is isomorphic to the anticanonical sheaf on  $X_\sigma$ . In particular, we have a generalized Calabi-Yau manifold  $Y_\sigma$  given by generic global sections of  $\mathcal{O}_{X_\sigma}(1)$ . There is an involution on the set of almost reflexive Gorenstein cones  $\sigma \mapsto \sigma^\bullet$ . For a dual pair of almost reflexive Gorenstein cones  $\sigma$  and  $\sigma^\bullet$  we expect that the correspondence between generalized Calabi-Yau manifolds  $Y_\sigma \leftrightarrow Y_{\sigma^\bullet}$  corresponds to the mirror involution in  $N = 2$  super conformal field theory.

## 1. $\mathbb{Q}$ -REFLEXIVE AND ALMOST REFLEXIVE POLYTOPES.

In this section, we first review the definition of reflexive polytopes due to V. Batyrev in [B1], and then construct a natural generalization of these notions for rational and lattice polytopes.

Let  $M$  be a lattice of rank  $d$  and  $N = \text{Hom}(M, \mathbb{Z})$  be its dual lattice with a natural paring  $\langle \_, \_ \rangle : M \times N \rightarrow \mathbb{Z}$ . Denote  $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ ,  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ .

**Definition 1.1.** A  $d$ -dimensional lattice polytope  $\Delta \subset M_{\mathbb{R}}$  is called a *canonical Fano polytope* if  $\text{int}(\Delta) \cap M = \{0\}$ .

**Definition 1.2.** [B1] A  $d$ -dimensional lattice polytope  $\Delta \subset M_{\mathbb{R}}$  is called *reflexive* (with respect to  $M$ ) if  $0 \in \text{int}(\Delta)$  and the dual polytope

$$\Delta^* = \{n \in N_{\mathbb{R}} \mid \langle m, n \rangle \geq -1 \forall m \in \Delta\}$$

in the dual vector space  $N_{\mathbb{R}}$  is also a lattice polytope. The pair  $\Delta$  and  $\Delta^*$  is called a *dual pair reflexive polytopes* and it satisfies  $\Delta = (\Delta^*)^*$ .

**Definition 1.3.** A compact toric variety  $X$  is called

- *Fano* if the anticanonical divisor  $-K_X$  is ample and  $\mathbb{Q}$ -Cartier,
- *Gorenstein* if  $K_X$  is Cartier.

**Proposition 1.4.** *There is a bijection between isomorphism classes of canonical Fano polytopes and Fano toric varieties with canonical singularities given by  $\Delta \mapsto X_{\Delta^*}$ . In particular, Gorenstein Fano toric varieties correspond to reflexive polytopes.*

Generalizing the notion of a reflexive polytope we introduce:

**Definition 1.5.** A  $d$ -dimensional polytope  $\Delta$  in  $M_{\mathbb{R}}$  is called  $\mathbb{Q}$ -*reflexive* (with respect to  $M$ ) if  $0 \in \text{int}(\Delta)$  and

$$\text{Conv}((\text{Conv}(\Delta \cap M))^* \cap N) = \Delta^*. \quad (1)$$

**Remark 1.6.** For a  $\mathbb{Q}$ -reflexive polytope  $\Delta$  its dual  $\Delta^*$  is a lattice polytope, whence reflexive polytopes are  $\mathbb{Q}$ -reflexive. It follows from (1) that a  $\mathbb{Q}$ -reflexive polytope is rational, i.e., its vertices lie in  $M_{\mathbb{Q}}$ . These properties together with the next ones suggest the name of  $\mathbb{Q}$ -reflexive.

**Definition 1.7.** Denote  $[\Delta] := \text{Conv}(\Delta \cap M)$  for a polytope  $\Delta$  in  $M_{\mathbb{R}}$  (and, similarly in  $N_{\mathbb{R}}$ ). Also, define  $\Delta^\circ := [\Delta]^* = (\text{Conv}(\Delta \cap M))^*$

In this notation, equation (1) is  $[[\Delta]^*] = \Delta^*$ , or, equivalently,  $(\Delta^\circ)^\circ = \Delta$ . Hence, we have

**Lemma 1.8.** *If  $\Delta \subset M_{\mathbb{R}}$  is  $\mathbb{Q}$ -reflexive, then  $\Delta^\circ = (\text{Conv}(\Delta \cap M))^* \subset N_{\mathbb{R}}$  is  $\mathbb{Q}$ -reflexive and the map  $\Delta \mapsto \Delta^\circ$  is an involution on the set of  $\mathbb{Q}$ -reflexive polytopes.*

We will call the pair of rational polytopes  $\Delta$  and  $\Delta^\circ$  as *the dual pair of  $\mathbb{Q}$ -reflexive polytopes*.

**Remark 1.9.** A  $\mathbb{Q}$ -reflexive polytope  $\Delta$  is completely determined by the convex hull  $[\Delta]$  of its lattice points since  $\Delta = [[\Delta]^*]^*$ .

**Definition 1.10.** A  $d$ -dimensional lattice polytope  $\Delta$  in  $N_{\mathbb{R}}$  is called *almost reflexive* (with respect to  $N$ ) if  $0 \in \text{int}(\Delta)$  and

$$\text{Conv}(\text{Conv}(\Delta^* \cap M))^* \cap N = \Delta. \quad (2)$$

**Lemma 1.11.** *A polytope  $\Delta$  in  $M_{\mathbb{R}}$  is  $\mathbb{Q}$ -reflexive if and only if  $\Delta^*$  in  $N_{\mathbb{R}}$  is almost reflexive. In particular, reflexive polytopes are almost reflexive.*

**Definition 1.12.** For a polytope  $\Delta$  in  $N_{\mathbb{R}}$  define  $\Delta^\bullet := [\Delta^*] = \text{Conv}(\Delta^* \cap M)$

**Remark 1.13.** In the new notation, equation (2) is  $[[\Delta^*]^*] = \Delta$ , or, equivalently,  $(\Delta^\bullet)^\bullet = \Delta$ .

**Lemma 1.14.** *If  $\Delta \subset N_{\mathbb{R}}$  is almost reflexive, then  $\Delta^\bullet := \text{Conv}(\Delta^* \cap M)$  is almost reflexive and the map  $\Delta \mapsto \Delta^\bullet$  is an involution on the set of almost reflexive polytopes.*

We will call the pair of lattice polytopes  $\Delta$  and  $\Delta^\bullet$  as *the dual pair of almost reflexive polytopes*.

A  $\mathbb{Q}$ -reflexive polytope has the following properties.

**Lemma 1.15.** *Every facet of a  $\mathbb{Q}$ -reflexive polytope contains a lattice point.*

*Proof.* Suppose that  $\Delta$  is  $\mathbb{Q}$ -reflexive. Since  $\Delta^*$  is a lattice polytope, every facet of  $\Delta$  is determined by  $\{m \in M_{\mathbb{R}} \mid \langle m, v \rangle = -1\}$  for a vertex  $v \in \Delta^*$ . If this facet does not contain a lattice point then  $\text{Conv}(\Delta \cap M)$  is contained in the half-space  $\{m \in M_{\mathbb{R}} \mid \langle m, v \rangle \geq 0\}$ . But then  $(\text{Conv}(\Delta \cap M))^*$  is unbounded, contradicting that  $\text{Conv}((\text{Conv}(\Delta \cap M))^* \cap N) = \Delta^*$  is a polytope.  $\square$

**Lemma 1.16.** *If  $\Delta$  is a  $\mathbb{Q}$ -reflexive polytope in  $M_{\mathbb{R}}$ , then*

- (a)  $\text{int}(\Delta) \cap M = \{0\}$ ,
- (b)  $\Delta^* = [\Delta^\circ]$ ,
- (c)  $\text{int}(\Delta^*) \cap N = \{0\}$ .

*Proof.* The property (a) follows from the fact that  $\Delta^*$  is a lattice polytope, while the property (b) is equation (1) in the new notation. Part (c) holds since  $\Delta^\circ$  is  $\mathbb{Q}$ -reflexive in  $N_{\mathbb{R}}$  and applying properties (a) and (b) to  $\Delta^\circ$  we get  $\text{int}(\Delta^*) \cap N = \text{int}(\Delta^\circ) \cap N = \{0\}$ .  $\square$

By part (c) of the above lemma, we get

**Corollary 1.17.** *If  $\Delta$  is a  $\mathbb{Q}$ -reflexive polytope, then  $\Delta^*$  is a canonical Fano polytope.*

## 2. $\mathbb{Q}$ -NEF-PARTITIONS.

In this section, we generalize the construction of nef-partitions of L. Borisov in [Bo] in the context of  $\mathbb{Q}$ -reflexive polytopes.

**Definition 2.1.** [Bo] A *nef-partition* of a reflexive polytope  $\Delta$  is a Minkowski sum decomposition  $\Delta = \Delta_1 + \cdots + \Delta_r$  by lattice polytopes such that  $0 \in \Delta_i$  for all  $i$ .

**Theorem 2.2.** [Bo] *Let  $\Delta = \Delta_1 + \cdots + \Delta_r$  be a nef-partition. If*

$$\nabla_j = \{y \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -\delta_{ij} \forall x \in \Delta_i, i = 1, \dots, r\}$$

*for  $j = 1, \dots, r$ , where  $\delta_{ij}$  is the Kronecker symbol, then  $\nabla = \nabla_1 + \cdots + \nabla_r$  is a nef-partition. Moreover,*

$$\Delta_i = \{x \in M_{\mathbb{R}} \mid \langle x, y \rangle \geq -\delta_{ij} \forall y \in \nabla_j, j = 1, \dots, r\}$$

*for  $i = 1, \dots, r$ .*

The nef-partitions  $\Delta = \Delta_1 + \cdots + \Delta_r$  and  $\nabla = \nabla_1 + \cdots + \nabla_r$  are called a *dual pair of nef-partitions*.

**Remark 2.3.** The name nef-partition comes from two words: nef and partition. The *nef* part comes from the property that each summand  $\Delta_i$  in the Minkowski sum  $\Delta = \Delta_1 + \cdots + \Delta_r$  defines a nef (numerically effective) divisor

$$D_{\Delta_i} = \sum_{\rho \in \Sigma_{\Delta}(1)} (-\min\langle \Delta_i, v_{\rho} \rangle) D_{\rho} = \sum_{v_{\rho} \in \nabla_i} D_{\rho}$$

on the Gorenstein Fano toric variety  $X_{\Delta}$ , where  $D_{\rho}$  are the torus invariant divisors in  $X_{\Delta}$  corresponding to the rays  $\rho$  of the normal fan  $\Sigma_{\Delta}$  of the polytope  $\Delta$ , and  $v_{\rho}$  are the primitive lattice generators of  $\rho$ . The *partition* part corresponds to the fact that the anticanonical divisor has its support  $\bigcup_{\rho \in \Sigma_{\Delta}(1)} D_{\rho}$  partitioned into the union of supports  $\bigcup_{v_{\rho} \in \nabla_i} D_{\rho}$  of the nef-divisors  $D_{\Delta_i}$ .

**Remark 2.4.** It was an original idea of Yu. I. Manin (see [BvS, Sect. 6.2]) to partition the disjoint union  $\bigcup_{\rho \in \Sigma_{\Delta}(1)} D_{\rho}$  of torus invariant divisors into a union of sets which support the nef-divisors  $D_{\Delta_i}$ . L. Borisov translated this idea into Minkowski sums and found a canonical way of creating dual nef-partitions.

Now, we introduce a generalization of nef-partition in the context of  $\mathbb{Q}$ -reflexive polytopes.

**Definition 2.5.** A  $\mathbb{Q}$ -nef-partition of a  $\mathbb{Q}$ -reflexive polytope  $\Delta$  is a Minkowski sum decomposition  $\Delta = \Delta_1 + \cdots + \Delta_r$  into polytopes in  $M_{\mathbb{R}}$  such that  $0 \in \Delta_i$  for all  $i$ , and  $\text{Conv}(\Delta \cap M) = \text{Conv}(\Delta_1 \cap M) + \cdots + \text{Conv}(\Delta_r \cap M)$ .

A  $\mathbb{Q}$ -nef-partition has the following property.

**Lemma 2.6.** Let  $\Delta = \Delta_1 + \cdots + \Delta_r$  be a  $\mathbb{Q}$ -nef-partition, and let  $F$  be a facet of  $\Delta$  and  $F = F_1 + \cdots + F_r$  be the induced decomposition by faces  $F_i$  of  $\Delta_i$ , for  $i = 1, \dots, r$ . Then  $\text{Conv}(F \cap M) = \text{Conv}(F_1 \cap M) + \cdots + \text{Conv}(F_r \cap M)$ .

*Proof.* Let  $F$  be a facet of  $\Delta$  with the induced decomposition  $F = F_1 + \cdots + F_r$ . Then the inclusion  $\text{Conv}(F_1 \cap M) + \cdots + \text{Conv}(F_r \cap M) \subseteq \text{Conv}(F \cap M)$  is clear. To show the other inclusion, notice that  $[F] = \text{Conv}(F \cap M)$  is a nonempty face of  $\text{Conv}(\Delta \cap M)$ , by Lemma 1.15. By Definition 2.5, we have  $[\Delta] = [\Delta_1] + \cdots + [\Delta_r]$ , which induces the Minkowski sum decomposition  $[F] = G_1 + \cdots + G_r$  by faces  $G_i$  of  $[\Delta_i]$ . Let  $v$  be the vertex of  $\Delta^*$  such that  $\langle F, v \rangle = \min \langle \Delta, v \rangle = -1$ . We have  $\min \langle \Delta_i, v \rangle \leq \min \langle [\Delta_i], v \rangle$  for all  $i$  and

$$\min \langle \Delta, v \rangle = \sum_{i=1}^r \min \langle \Delta_i, v \rangle \leq \sum_{i=1}^r \min \langle [\Delta_i], v \rangle = \min \langle [\Delta], v \rangle. \quad (3)$$

Hence,  $\min \langle \Delta_i, v \rangle = \min \langle [\Delta_i], v \rangle$  for all  $i$ , since  $\min \langle \Delta, v \rangle = \min \langle [\Delta], v \rangle = -1$  by Lemma 1.15. Since the faces  $F_i$  and  $G_i$  are determined by the minimal value of  $v$  on  $\Delta_i$  and  $[\Delta_i]$ , respectively, we conclude that  $G_i \subseteq F_i$ , whence  $[F] = G_1 + \cdots + G_r \subseteq \text{Conv}(F_1 \cap M) + \cdots + \text{Conv}(F_r \cap M)$ .  $\square$

**Definition 2.7.** For a  $\mathbb{Q}$ -nef-partition  $\Delta_1 + \cdots + \Delta_r$  in  $M_{\mathbb{R}}$  define the polytopes

$$\nabla_j = \{y \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -\delta_{ij} \forall x \in \text{Conv}(\Delta_i \cap M), i = 1, \dots, r\} \quad (4)$$

for  $j = 1, \dots, r$ .

**Proposition 2.8.** Let  $\Delta_1 + \cdots + \Delta_r$  be a  $\mathbb{Q}$ -nef-partition in  $M_{\mathbb{R}}$ , then

$$(\Delta_1 + \cdots + \Delta_r)^* = \text{Conv}(\nabla_1 \cap N, \dots, \nabla_r \cap N),$$

where  $\nabla_1, \dots, \nabla_r$  are defined by (4).

*Proof.* Let  $v$  be a vertex of  $\Delta^*$ , where  $\Delta = \Delta_1 + \cdots + \Delta_r$  is a  $\mathbb{Q}$ -nef-partition. Then by (3) and Lemma 1.15, we have  $\sum_{i=1}^r \min \langle [\Delta_i], v \rangle = -1$ , whence the integer  $\min \langle [\Delta_j], v \rangle = -1$  for some  $j$  and  $\min \langle [\Delta_i], v \rangle = 0$  for  $i \neq j$  since  $\min \langle [\Delta_i], v \rangle \leq 0$  by  $0 \in [\Delta_i]$  for all  $i$ . Hence, every vertex  $v$  of  $\Delta^*$  is contained in some  $\nabla_j \cap N$ , and  $\Delta^* \subseteq \text{Conv}(\nabla_1 \cap N, \dots, \nabla_r \cap N)$ .

To show the opposite inclusion, let  $y \in \nabla_j \cap N$  for some  $j$ . Then  $\min \langle [\Delta], y \rangle = \sum_{i=1}^r \min \langle [\Delta_i], y \rangle \geq \sum_{i=1}^r \delta_{ij} = -1$ , whence  $y \in [\Delta]^* = \Delta^\circ$ . Since  $y$  is a lattice point, we get  $y \in [\Delta^\circ] = \Delta^*$  by part (b) of Lemma 1.16.  $\square$

**Proposition 2.9.** Let  $\Delta_1 + \cdots + \Delta_r$  be a  $\mathbb{Q}$ -nef-partition in  $M_{\mathbb{R}}$ , then

$$(\nabla_1 + \cdots + \nabla_r)^* = \text{Conv}(\Delta_1 \cap M, \dots, \Delta_r \cap M),$$

where  $\nabla_1, \dots, \nabla_r$  are defined by (4).

*Proof.* One inclusion  $\text{Conv}([\Delta_1], \dots, [\Delta_r]) \subseteq (\nabla_1 + \cdots + \nabla_r)^*$  holds since  $\langle [\Delta_i], \nabla_j \rangle \geq -\delta_{ij}$  for all  $i, j$  by (4).

The opposite inclusion holds because  $[\Delta_i]$  are lattice polytopes with  $0 \in [\Delta_i]$  and 0 is the only interior lattice point in  $[\Delta_1] + \cdots + [\Delta_r]$  by Definition 2.5 and part (a) of Lemma 1.16  $\square$

**Definition 2.10.** Let  $P_1, \dots, P_r$  be polytopes in  $M_{\mathbb{R}}$ . Consider the lattice  $\bar{M} = M \oplus \mathbb{Z}^r$ , where  $\{e_1, \dots, e_r\}$  is the standard basis of  $\mathbb{Z}^r$ . The cone

$$\mathcal{C}_{P_1, \dots, P_r} := \mathbb{R}_{\geq 0} \cdot \text{Conv}(P_1 + e_1, \dots, P_r + e_r)$$

is called the *Cayley cone* associated to the  $r$ -tuple of polytopes  $P_1, \dots, P_r$ .

**Lemma 2.11.** [M4, Lem. 1.6] *Let  $P_1, \dots, P_r$  be polytopes in  $M_{\mathbb{R}}$  such that  $P = P_1 + \dots + P_r$  is  $d$ -dimensional and  $0 \in \text{int}(P)$ . Then the dual of the Cayley cone associated to  $P_1, \dots, P_r$  is*

$$\mathcal{C}_{P_1, \dots, P_r}^{\vee} = \mathbb{R}_{\geq 0} \cdot \text{Conv}\left(\left\{x - \sum_{i=1}^r \min\langle \Delta_i, x \rangle e_i^* \mid x \in \mathcal{V}(P^*)\right\}, \{e_1^*, \dots, e_r^*\}\right),$$

where  $\mathcal{V}(P^*)$  is the set of vertices of  $P^*$  and  $\{e_1^*, \dots, e_r^*\}$  is the basis of  $\mathbb{Z}^r \subset N \oplus \mathbb{Z}^r$  dual to  $\{e_1, \dots, e_r\}$ .

Using this lemma from Propositions 2.8 and 2.9 we get

**Proposition 2.12.** *Let  $\Delta_1 + \dots + \Delta_r$  be a  $\mathbb{Q}$ -nef-partition in  $M_{\mathbb{R}}$ , then*

$$\mathcal{C}_{\Delta_1, \dots, \Delta_r}^{\vee} = \mathcal{C}_{[\nabla_1], \dots, [\nabla_r]}, \quad \mathcal{C}_{\nabla_1, \dots, \nabla_r}^{\vee} = \mathcal{C}_{[\Delta_1], \dots, [\Delta_r]},$$

where  $\nabla_1, \dots, \nabla_r$  are defined by (4).

Applying a projection technique from [BN] to the Cayley cones in the last proposition we get the following one.

**Proposition 2.13.** *Let  $\Delta_1 + \dots + \Delta_r$  be a  $\mathbb{Q}$ -nef-partition in  $M_{\mathbb{R}}$ , then*

$$\begin{aligned} (\text{Conv}(\Delta_1, \dots, \Delta_r))^* &= \text{Conv}(\nabla_1 \cap N) + \dots + \text{Conv}(\nabla_r \cap N), \\ (\text{Conv}(\nabla_1, \dots, \nabla_r))^* &= \text{Conv}(\Delta_1 \cap M) + \dots + \text{Conv}(\Delta_r \cap M), \end{aligned}$$

where  $\nabla_1, \dots, \nabla_r$  are defined by (4).

**Proposition 2.14.** *Let  $\Delta_1 + \dots + \Delta_r$  be a  $\mathbb{Q}$ -nef-partition in  $M_{\mathbb{R}}$ , and let  $\nabla_j$  be defined by (4). Then  $\text{Conv}(\nabla_1, \dots, \nabla_r)$  is a  $\mathbb{Q}$ -reflexive polytope and*

$$(\Delta_1 + \dots + \Delta_r)^{\circ} = \text{Conv}(\nabla_1, \dots, \nabla_r).$$

*Proof.* By Definitions 1.7, 2.5, and Proposition 2.13 we have

$$(\Delta_1 + \dots + \Delta_r)^{\circ} = [\Delta_1 + \dots + \Delta_r]^* = ([\Delta_1] + \dots + [\Delta_r])^* = \text{Conv}(\nabla_1, \dots, \nabla_r).$$

□

**Proposition 2.15.** *Let  $\Delta_1 + \dots + \Delta_r$  be a  $\mathbb{Q}$ -nef-partition in  $M_{\mathbb{R}}$ , and let  $\nabla_1, \dots, \nabla_r$  be defined by (4). Then  $\nabla_1 + \dots + \nabla_r$  and  $\text{Conv}(\Delta_1, \dots, \Delta_r)$  are  $\mathbb{Q}$ -reflexive polytopes and*

$$(\nabla_1 + \dots + \nabla_r)^{\circ} = \text{Conv}(\Delta_1, \dots, \Delta_r).$$

*Proof.* First, we claim

$$\text{Conv}([\Delta_1], \dots, [\Delta_r]) = [\text{Conv}(\Delta_1, \dots, \Delta_r)]. \quad (5)$$

Indeed, take  $x \in \text{Conv}(\Delta_1, \dots, \Delta_r) \cap M$ , then  $x \in (\sum_{i=1}^r [\nabla_i])^*$  by Proposition 2.13, and we have

$$-1 \leq \min\langle x, \sum_{i=1}^r [\nabla_i] \rangle = \sum_{i=1}^r \min\langle x, [\nabla_i] \rangle \leq 0,$$

since  $0 \in [\nabla_i]$  for all  $i$ . From here we get two cases: either all integers  $\min\langle x, [\nabla_i] \rangle = 0$ , for  $i = 1, \dots, r$ , or there is  $j$  such that  $\min\langle x, [\nabla_i] \rangle = -\delta_{ij}$  for  $i = 1, \dots, r$ . This implies that either  $x = 0$  or  $x \in \Delta_j$ . Thus, we showed that (5) holds.

Next, we show that  $\nabla_1 + \dots + \nabla_r$  is  $\mathbb{Q}$ -reflexive. Using Propositions 2.9, 2.13, we have

$$\begin{aligned} \text{Conv}([\Delta_1], \dots, [\Delta_r]) &= (\nabla_1 + \dots + \nabla_r)^* \subseteq [\nabla_1 + \dots + \nabla_r]^* \subseteq \\ &\subseteq ([\nabla_1] + \dots + [\nabla_r])^* = \text{Conv}(\Delta_1, \dots, \Delta_r). \end{aligned}$$

Applying (5) to this, we get

$$[[\nabla_1 + \dots + \nabla_r]^*] = \text{Conv}([\Delta_1], \dots, [\Delta_r]) = (\nabla_1 + \dots + \nabla_r)^*,$$

showing that the polytope  $\nabla_1 + \dots + \nabla_r$  is  $\mathbb{Q}$ -reflexive. Then, by Definition 1.7 and the properties of  $\mathbb{Q}$ -reflexive polytopes, the dual  $\mathbb{Q}$ -reflexive polytope is  $(\nabla_1 + \dots + \nabla_r)^\circ = \text{Conv}(\Delta_1, \dots, \Delta_r)$ .  $\square$

Finally, we establish the existence of the dual  $\mathbb{Q}$ -nef-partition:

**Theorem 2.16.** *Let  $\Delta_1 + \dots + \Delta_r$  be a  $\mathbb{Q}$ -nef-partition, then  $\nabla_1 + \dots + \nabla_r$  is a  $\mathbb{Q}$ -nef-partition, where  $\nabla_1, \dots, \nabla_r$  are defined by (4). Moreover,*

$$\Delta_i = \{x \in M_{\mathbb{R}} \mid \langle x, y \rangle \geq -\delta_{ij} \forall y \in \text{Conv}(\nabla_j \cap N), j = 1, \dots, r\}.$$

*Proof.* Proposition 2.15 gives  $\mathbb{Q}$ -reflexivity of  $\nabla_1 + \dots + \nabla_r$ . To show that  $\nabla_1 + \dots + \nabla_r$  is a  $\mathbb{Q}$ -nef-partition notice

$$(\text{Conv}(\Delta_1, \dots, \Delta_r))^* = [\nabla_1] + \dots + [\nabla_r] \subseteq [\nabla_1 + \dots + \nabla_r]$$

by Proposition 2.13. Applying part (b) of Lemma 1.16, we see that  $[\nabla_1 + \dots + \nabla_r] = (\text{Conv}(\Delta_1, \dots, \Delta_r))^*$  since  $\text{Conv}(\Delta_1, \dots, \Delta_r) = (\nabla_1 + \dots + \nabla_r)^\circ$  by Proposition 2.15. From the above inclusions we get the required equality  $[\nabla_1 + \dots + \nabla_r] = [\nabla_1] + \dots + [\nabla_r]$  in the definition of a  $\mathbb{Q}$ -nef-partition. The last part of this theorem follows by Proposition 2.8 since 0 is the only interior lattice point in  $[\nabla_1] + \dots + [\nabla_r]$ .  $\square$

The Minkowski sums  $\Delta_1 + \dots + \Delta_r$  and  $\nabla_1 + \dots + \nabla_r$  in Theorem 2.16 will be called a *dual pair of  $\mathbb{Q}$ -nef-partitions*.

**Definition 2.17.** A  $\mathbb{Q}$ -nef-partition  $\Delta_1 + \dots + \Delta_r$  in  $M_{\mathbb{R}}$  is called *proper* if  $\Delta_i \neq 0$  for all  $1 \leq i \leq r$ .

**Corollary 2.18.** *Let  $\Delta_1 + \dots + \Delta_r \subset M_{\mathbb{R}}$  and  $\nabla_1 + \dots + \nabla_r \subset N_{\mathbb{R}}$  be a dual pair of  $\mathbb{Q}$ -nef-partitions. Then, for  $i = 1, \dots, r$ , one has  $\Delta_i = 0$  if and only if  $\nabla_i = 0$ .*

*Proof.* If  $\Delta_i = 0$ , then  $\nabla_i = 0$  by Definition 2.5, since  $[\Delta_1] + \dots + [\Delta_r]$  spans  $M_{\mathbb{R}}$ . The opposite implication follows from Theorem 2.16.  $\square$

**Corollary 2.19.** *If  $\Delta_1 + \dots + \Delta_r \subset M_{\mathbb{R}}$  is a proper  $\mathbb{Q}$ -nef-partition, then its dual  $\mathbb{Q}$ -nef-partition  $\nabla_1 + \dots + \nabla_r$  is proper.*

**Corollary 2.20.** *If  $\Delta_1 + \dots + \Delta_r \subset M_{\mathbb{R}}$  is a proper  $\mathbb{Q}$ -nef-partition, then  $\Delta_i \cap M \neq \{0\}$  for all  $1 \leq i \leq r$ .*



## 3. ALMOST REFLEXIVE GORENSTEIN CONES.

In this section, we generalize the notion of reflexive Gorenstein cones.

**Definition 3.1.** [BBo1] Let  $\bar{M}$  and  $\bar{N}$  be a pair of dual lattices of rank  $\bar{d}$ . A  $\bar{d}$ -dimensional polyhedral cone  $\sigma$  with a vertex at  $0 \in \bar{M}$  is called *Gorenstein*, if it is generated by finitely many lattice points contained in the affine hyperplane  $\{x \in \bar{M} \mid \langle x, h_\sigma \rangle = 1\}$  for  $h_\sigma \in \bar{N}$ . The unique lattice point  $h_\sigma$  is called the *height* (or *degree*) vector of the Gorenstein cone  $\sigma$ . A Gorenstein cone  $\sigma$  is called *reflexive* if both  $\sigma$  and its dual

$$\sigma^\vee = \{y \in \bar{N}_\mathbb{R} \mid \langle x, y \rangle \geq 0 \forall x \in \sigma\}$$

are Gorenstein cones. In this case, they both have uniquely determined  $h_\sigma \in \bar{N}$  and  $h_{\sigma^\vee} \in \bar{M}$ , which take value 1 at the primitive lattice generators of the respective cones. The positive integer  $r = \langle h_{\sigma^\vee}, h_\sigma \rangle$  is called the *index* of the reflexive Gorenstein cones  $\sigma$  and  $\sigma^\vee$ .

As in [BN], denote  $\sigma_{(i)} := \{x \in \sigma \mid \langle x, h_\sigma \rangle = i\}$ , for  $i \in \mathbb{N}$ . The basic relationship between reflexive polytopes and reflexive Gorenstein cones is provided by the following:

**Proposition 3.2.** [BBo1, Pr. 2.11] *Let  $\sigma$  be a Gorenstein cone. Then  $\sigma$  is a reflexive Gorenstein cone of index  $r$  if and only if the polytope  $\sigma_{(r)} - h_{\sigma^\vee}$  is a reflexive polytope with respect to the lattice  $\bar{M} \cap h_\sigma^\perp = \{x \in \bar{M} \mid \langle x, h_\sigma \rangle = 0\}$ .*

Generalizing the notion of reflexive Gorenstein cones we introduce:

**Definition 3.3.** A Gorenstein cone  $\sigma$  in  $\bar{M}_\mathbb{R}$  is called *almost reflexive*, if there is  $r \in \mathbb{N}$  such that  $\sigma_{(r)}$  has a unique lattice point  $h$  in its relative interior and  $\sigma_{(r)} - h$  is an almost reflexive polytope with respect to the lattice  $\bar{M} \cap h_\sigma^\perp$ . We will denote  $h$  by  $h_{\sigma^\vee}$ . The positive integer  $r$  will be called the *index* of the almost reflexive Gorenstein cone  $\sigma$ .

**Lemma 3.4.** *Reflexive Gorenstein cones are almost reflexive.*

*Proof.* This follows from Proposition 3.2 and Lemma 1.11.  $\square$

**Definition 3.5.** For an almost reflexive Gorenstein cone  $\sigma$  in  $\bar{M}_\mathbb{R}$  define

$$\sigma_{(i)}^\vee = \{y \in \sigma^\vee \mid \langle h_{\sigma^\vee}, y \rangle = i\}, \text{ for } i \in \mathbb{N}.$$

Denote  $[\sigma^\vee] := \mathbb{R}_{\geq 0}[\sigma_{(1)}^\vee] = \mathbb{R}_{\geq 0}\text{Conv}(\sigma_{(1)}^\vee \cap \bar{M})$

**Lemma 3.6.** *Let  $\Delta$  be a polytope in  $M_\mathbb{R}$  with  $0 \in \text{int}(\Delta)$ , and let  $\sigma_\Delta = \mathbb{R}_{\geq 0}(\Delta, 1) \subset \bar{M}_\mathbb{R} = M_\mathbb{R} \oplus \mathbb{R}$ . Then*

$$\sigma_\Delta^\vee = \sigma_{\Delta^*} = \mathbb{R}_{\geq 0}(\Delta^*, 1) \subset \bar{N}_\mathbb{R} = N_\mathbb{R} \oplus \mathbb{R}.$$

**Corollary 3.7.** *A Gorenstein cone  $\sigma$  in  $\bar{M}_\mathbb{R}$  is almost reflexive of index 1 if and only if the polytope  $\sigma_{(1)}^\vee - h_\sigma$  is  $\mathbb{Q}$ -reflexive with respect to the lattice  $\bar{N} \cap h_\sigma^\perp$ .*

*Proof.* Combine Lemmas 1.11 and 3.6.  $\square$

**Corollary 3.8.** *If  $\sigma$  in  $\bar{M}_\mathbb{R}$  is an almost reflexive Gorenstein cone of index 1, then  $[\sigma^\vee]$  is an almost reflexive Gorenstein cone of index 1.*

**Proposition 3.9.** *If  $\sigma$  in  $\bar{M}_\mathbb{R}$  is an almost reflexive Gorenstein cone of index  $r$ , then  $[\sigma^\vee]$  is an almost reflexive Gorenstein cone of index  $r$ .*

*Proof.* Use the techniques in the proof of [BBo1, Pr. 2.11].  $\square$

Almost reflexive Gorenstein cones have the following property.

**Lemma 3.10.** *Let  $\sigma \subset \bar{M}_{\mathbb{R}}$  be an almost reflexive Gorenstein cone. Then  $[[\sigma^{\vee}]^{\vee}] = \sigma$ .*

**Definition 3.11.** For an almost reflexive Gorenstein cone  $\sigma$ , denote  $\sigma^{\bullet} := [\sigma^{\vee}]$ .

**Corollary 3.12.** *The map  $\sigma \mapsto \sigma^{\bullet}$  is an involution on the set of almost reflexive Gorenstein cones:  $(\sigma^{\bullet})^{\bullet} = \sigma$ .*

Cayley cones corresponding to a dual pair of  $\mathbb{Q}$ -nef-partitions are related to almost reflexive Gorenstein cones as follows:

**Proposition 3.13.** *Let  $\Delta_1 + \dots + \Delta_r \subset M_{\mathbb{R}}$  and  $\nabla_1 + \dots + \nabla_r \subset N_{\mathbb{R}}$  be a dual pair of  $\mathbb{Q}$ -nef-partitions. Then the Cayley cones  $\mathcal{C}_{[\Delta_1], \dots, [\Delta_r]}$  and  $\mathcal{C}_{[\nabla_1], \dots, [\nabla_r]}$  is a dual pair of almost reflexive Gorenstein cones:*

$$\mathcal{C}_{[\Delta_1], \dots, [\Delta_r]}^{\bullet} = [\mathcal{C}_{[\Delta_1], \dots, [\Delta_r]}^{\vee}] = \mathcal{C}_{[\nabla_1], \dots, [\nabla_r]}.$$

*Proof.* This follows directly from Proposition 2.12 since the height vectors of the Cayley cones  $\mathcal{C}_{[\Delta_1], \dots, [\Delta_r]}$  and  $\mathcal{C}_{[\nabla_1], \dots, [\nabla_r]}$  are  $e_1^* + \dots + e_r^*$  and  $e_1 + \dots + e_r$ , respectively.  $\square$

#### 4. BASIC TORIC GEOMETRY.

This section will review some basics of toric geometry.

Let  $X_{\Sigma}$  be a  $d$ -dimensional toric variety associated with a finite rational polyhedral fan  $\Sigma$  in  $N_{\mathbb{R}}$ . Denote by  $\Sigma(1)$  the finite set of the 1-dimensional cones  $\rho$  in  $\Sigma$ , which correspond to the torus invariant divisors  $D_{\rho}$  in  $X_{\Sigma}$ . By [C], every toric variety can be described as a categorical quotient of a Zariski open subset of an affine space by a subgroup of a torus. Consider the polynomial ring  $S(\Sigma) := \mathbb{C}[x_{\rho} : \rho \in \Sigma(1)]$ , called the *homogeneous coordinate ring* of the toric variety  $X_{\Sigma}$ , and the corresponding affine space  $\mathbb{C}^{\Sigma(1)} = \text{Spec}(\mathbb{C}[x_{\rho} : \rho \in \Sigma(1)])$ . The ideal  $B = \langle \prod_{\rho \notin \sigma} x_{\rho} : \sigma \in \Sigma \rangle$  in  $S$  is called the irrelevant ideal. This ideal determines a Zariski closed set  $\mathbf{V}(B)$  in  $\mathbb{C}^{\Sigma(1)}$ , which is invariant under the diagonal group action of the subgroup

$$G = \left\{ (\mu_{\rho}) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \prod_{\rho \in \Sigma(1)} \mu_{\rho}^{\langle u, v_{\rho} \rangle} = 1 \forall u \in M \right\}$$

of the torus  $(\mathbb{C}^*)^{\Sigma(1)}$  on the affine space  $\mathbb{C}^{\Sigma(1)}$ , where  $v_{\rho}$  denotes the primitive lattice generator of the 1-dimensional cone  $\rho$ . The toric variety  $X_{\Sigma}$  is isomorphic to the categorical quotient  $(\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B))/G$ , which is induced by a toric morphism  $\pi : \mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B) \rightarrow X_{\Sigma}$ , constant on  $G(\Sigma)$ -orbits (see [CLS, Thm. 5.1.10]).

The coordinate ring  $S(\Sigma)$  is graded by the Chow group

$$A_{d-1}(X_{\Sigma}) \simeq \text{Hom}(G, \mathbb{C}^*),$$

and  $\deg(\prod_{\rho \in \Sigma(1)} x_{\rho}^{b_{\rho}}) = [\sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}] \in A_{d-1}(X_{\Sigma})$ . For a torus invariant Weil divisor  $D = \sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}$ , there is a one-to-one correspondence between the monomials of  $\mathbb{C}[x_{\rho} : \rho \in \Sigma(1)]$  in the degree  $[\sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho}] \in A_{d-1}(X_{\Sigma})$  and the lattice points inside the polytope

$$\Delta_D = \{m \in M_{\mathbb{R}} \mid \langle m, v_{\rho} \rangle \geq -b_{\rho} \forall \rho \in \Sigma(1)\}$$

by associating to  $m \in \Delta_D$  the monomial  $\prod_{\rho \in \Sigma(1)} x_\rho^{b_\rho + \langle m, v_\rho \rangle}$ . If we denote the homogeneous degree of  $S(\Sigma)$  corresponding to  $\beta = [D] \in A_{d-1}(X_\Sigma)$  by  $S(\Sigma)_\beta$ , then by [C, Prop. 1.1], we also have a natural isomorphism

$$H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \simeq S(\Sigma)_\beta.$$

In particular, every hypersurface in  $X_\Sigma$  of degree  $\beta = \sum_{\rho \in \Sigma(1)} b_\rho D_\rho$  corresponds to a polynomial

$$\sum_{m \in \Delta_D \cap M} a_m \prod_{\rho \in \Sigma(1)} x_\rho^{b_\rho + \langle m, v_\rho \rangle}$$

with the coefficients  $a_m \in \mathbb{C}$ . By [CLS, Prop. 5.2.8], all closed subvarieties of  $X_\Sigma$  correspond to homogeneous ideals  $I \subseteq S(\Sigma)$ , and [M3, Thm. 1.2] shows that a closed subvariety in a toric variety  $X_\Sigma$  can be viewed as a categorical quotient as well. A *complete intersection* in the toric variety  $X_\Sigma$  (in homogeneous coordinates) is a closed subvariety  $\mathbf{V}(I) \subset X_\Sigma$  corresponding to a radical homogeneous ideal  $I \subseteq S(\Sigma)$  generated by a regular sequence of homogeneous polynomials  $f_1, \dots, f_k \in S(\Sigma)$  such that  $k = \dim X_\Sigma - \dim \mathbf{V}(I)$  (see [M3, Sect. 1]).

Every rational polytope  $\Delta$  in  $M_\mathbb{R}$  determines the Weil  $\mathbb{Q}$ -divisor

$$D_\Delta = \sum_{\rho \in \Sigma(1)} (-\min\langle \Delta, v_\rho \rangle) D_\rho \in \text{WDiv}(X_\Sigma) \otimes_\mathbb{Z} \mathbb{Q}$$

on  $X_\Sigma$ .

**Definition 4.1.** Let  $X$  be a complete variety. A  $\mathbb{Q}$ -Cartier divisor  $D \in \text{Div}(X) \otimes_\mathbb{Z} \mathbb{Q}$  on  $X$  is called *nef* (numerically effective) if  $D \cdot C \geq 0$  for all irreducible curves  $C \subset X$ . We will call such divisors  $\mathbb{Q}$ -*nef*.

**Lemma 4.2.** *Let  $X_\Sigma$  be a compact toric variety. Then the divisor  $D_\Delta$  is  $\mathbb{Q}$ -nef if and only if its support function  $\psi_\Delta = -\min\langle \Delta, \_ \rangle$  is convex piecewise linear with respect to the fan  $\Sigma$ .*

*Proof.* By [F, p. 68], we know that  $\mathcal{O}_{X_\Sigma}(nD_\Delta)$  is generated by global sections for some sufficiently large  $n \in \mathbb{N}$  if and only if  $\psi_\Delta$  is convex piecewise linear on  $\Sigma$ . On the other hand, we showed in [M1, Thm. 1.6] that, for a compact toric variety  $X_\Sigma$ , the invertible sheaf  $\mathcal{O}_{X_\Sigma}(D)$  is generated by global sections if and only if  $D$  is nef.  $\square$

**Lemma 4.3.** [M3, Lem 2.1] *Let  $X_\Sigma$  be a compact toric variety associated to a fan  $\Sigma$  in  $N_\mathbb{R}$ . Suppose  $\Delta_1$  and  $\Delta_2$  are rational polytopes in  $M_\mathbb{R}$  then  $D_{\Delta_1 + \Delta_2}$  is a  $\mathbb{Q}$ -nef divisor on  $X_\Sigma$  iff  $D_{\Delta_1}$  and  $D_{\Delta_2}$  are  $\mathbb{Q}$ -nef on  $X_\Sigma$ .*

Every rational polytope  $\Delta$  in  $M_\mathbb{R}$  corresponds to a projective toric variety  $X_\Delta := X_{\Sigma_\Delta}$ , whose fan  $\Sigma_\Delta$  (called the *normal fan of  $\Delta$* ) is the collection of cones

$$\sigma_F = \{y \in N_\mathbb{R} \mid \langle x, y \rangle \leq \min\langle \Delta, y \rangle \forall x \in F\}.$$

The support function  $\psi_\Delta = -\min\langle \Delta, \_ \rangle$  is strictly convex piecewise linear with respect to the fan  $\Sigma_\Delta$ . In this case, the divisor  $D_\Delta$  is ample, and, in particular,  $\mathbb{Q}$ -nef. From Lemma 4.3, we get

**Corollary 4.4.** *Let  $X_\Delta$  be a Fano toric variety, and suppose  $\Delta = \Delta_1 + \dots + \Delta_r$  is a Minkowski sum decomposition by rational polytopes. Then the divisors  $D_{\Delta_i}$  are  $\mathbb{Q}$ -nef on  $X_\Delta$  for all  $1 \leq i \leq r$ .*

There is an alternative way to describe projective toric varieties using the Proj functor, which is simple but less useful in the context of complete intersections. Consider the cone

$$K = \{(t\Delta, t) \mid t \in \mathbb{R}_{\geq 0}\} \subset M_{\mathbb{R}} \oplus \mathbb{R}.$$

The projective toric variety  $X_{\Delta} = X_{\Sigma_{\Delta}}$  can be represented as  $\text{Proj}(\mathbb{C}[K \cap (M \oplus \mathbb{Z})])$ . Moreover, if  $\beta \in A_{d-1}(X_{\Delta})$  is the class of the ample divisor  $D_{\Delta} = \sum_{\rho \in \Sigma_{\Delta}(1)} b_{\rho} D_{\rho}$ , then there is a natural isomorphism of graded rings

$$\mathbb{C}[K \cap (M \oplus \mathbb{Z})] \simeq \bigoplus_{i=0}^{\infty} S(\Sigma_{\Delta})_{i\beta},$$

sending  $\chi^{(m,i)} \in \mathbb{C}[K \cap (M \oplus \mathbb{Z})]_i$  to  $\prod_{\rho \in \Sigma_{\Delta}(1)} x_{\rho}^{ib_{\rho} + \langle m, v_{\rho} \rangle}$ . In particular, a hypersurface given by a polynomial in homogeneous coordinates

$$\sum_{m \in \Delta \cap M} a_m \prod_{\rho \in \Sigma_{\Delta}(1)} x_{\rho}^{ib_{\rho} + \langle m, v_{\rho} \rangle} = 0$$

corresponds to  $\sum_{m \in \Delta \cap M} a_m \chi^{(m,i)} = 0$ .

## 5. MIRROR SYMMETRY CONSTRUCTION.

In this section, we propose a generalization of the Batyrev-Borisov Mirror Symmetry constructions.

A proper  $\mathbb{Q}$ -nef-partition  $\Delta = \Delta_1 + \dots + \Delta_r$  with  $r < d$  defines a  $\mathbb{Q}$ -nef Calabi-Yau complete intersection  $Y_{\Delta_1, \dots, \Delta_r}$  in the Fano toric variety  $X_{\Delta}$  given by the equations:

$$\left( \sum_{m \in \Delta_i \cap M} a_{i,m} \prod_{v_{\rho} \in \mathcal{V}(\Delta^*)} x_{\rho}^{\langle m, v_{\rho} \rangle} \right) \prod_{v_{\rho} \in [\nabla_i]} x_{\rho} = 0, \quad i = 1, \dots, r,$$

where  $x_{\rho}$  are the homogeneous coordinates of the toric variety  $X_{\Delta}$  corresponding to the vertices  $v_{\rho}$  of the polytope  $\Delta^*$ .

Following the Batyrev-Borisov Mirror Symmetry construction, we naturally expect that Calabi-Yau complete intersections corresponding to a dual pair of  $\mathbb{Q}$ -nef-partitions pass the topological mirror symmetry test:

**Conjecture 5.1.** *Let  $Y_{\Delta_1, \dots, \Delta_r} \subset X_{\Delta}$  and  $Y_{\nabla_1, \dots, \nabla_r} \subset X_{\nabla}$  be a pair of generic Calabi-Yau complete intersections in  $d$ -dimensional Fano toric varieties corresponding to a dual pair of  $\mathbb{Q}$ -nef-partitions  $\Delta = \Delta_1 + \dots + \Delta_r$  and  $\nabla = \nabla_1 + \dots + \nabla_r$ . Then*

$$h_{\text{st}}^{p,q}(Y_{\Delta_1, \dots, \Delta_r}) = h_{\text{st}}^{d-r-p,q}(Y_{\nabla_1, \dots, \nabla_r}), \quad 0 \leq p, q \leq d-r.$$

Assuming that this conjecture holds, by taking maximal projective crepant partial resolutions we obtain the mirror pair of minimal Calabi-Yau complete intersections  $\hat{Y}_{\Delta_1, \dots, \Delta_k}, \hat{Y}_{\nabla_1, \dots, \nabla_k}$ .

For an almost reflexive Gorenstein cone  $\sigma$  in  $\bar{M}_{\mathbb{R}}$ , we have the Fano toric variety  $X_{\sigma} = \text{Proj}(\mathbb{C}[\sigma^{\vee} \cap \bar{N}])$ , whose fan consists of cones generated by the faces of the almost reflexive polytope  $\sigma_{(r)} - h_{\sigma^{\vee}} \in \bar{M}_{\mathbb{R}} \cap h_{\sigma}^{\perp}$ . A *generalized Calabi-Yau manifold* is defined as the ample  $\mathbb{Q}$ -Cartier hypersurface  $Y_{\sigma} \subset X_{\sigma}$  given by the equation

$$\sum_{n \in \sigma_{(1)}^{\bullet} \cap \bar{N}} a_n \chi^n = 0$$

with generic  $a_n \in \mathbb{C}$ , where  $\chi^n$  are the elements in the graded semigroup ring  $\mathbb{C}[\sigma^\vee \cap \bar{N}]$  corresponding to the lattice points  $n \in \sigma^\vee \cap \bar{N}$ .

**Conjecture 5.2.** *The involution  $\sigma \mapsto \sigma^\bullet$  on the set of almost reflexive Gorenstein cones corresponds to the mirror involution of  $N = 2$  super conformal field theories associated to the generalized Calabi-Yau manifolds  $Y_\sigma$  and  $Y_{\sigma^\bullet}$ .*

In the case, when a  $\mathbb{Q}$ -nef Calabi-Yau complete intersection  $Y_{\Delta_1, \dots, \Delta_r}$  does not have the property that  $0 \in \Delta_i$  for all  $1 \leq i \leq r$  (i.e, the Minkowski sum  $\Delta_1 + \dots + \Delta_r$  is not a  $\mathbb{Q}$ -nef-partition), one can still associate to it the mirror in the form of the generalized Calabi-Yau manifold corresponding to the dual of the Cayley cone  $\mathcal{C}_{\Delta_1, \dots, \Delta_r}$ .

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